Mom! There's an Astroid in My Closet!

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The knowledge of the world is only to be acquired in the world, and not in a closet.

—Lord Chesterfield, from Letters to His Son (1694–1773)

Introduction

All children know that there are mysteries, sometimes frightening mysteries, hidden in closets. Adults often brush this aside as the result of an overactive imagination. But perhaps we should take a second look. Perhaps it is *our* imaginations that are *under*active. If you have a closet (or any doorway) covered with a bifold door there is an astroid lurking just inside and the only way you can get to it is to coax it carefully with a little bit of calculus. If your door has more than one fold there are even more interesting objects waiting to be discovered.

This investigation began when one of the authors (Seiple) was standing at his closet wondering how much floor space was needed to accommodate the opening and closing of the bifold door mounted on it. He was supposed to be getting dressed for school, but he was in high school at the time so perhaps he can be forgiven. When he arrived at college he described the problem to Boman and Brazier who encouraged him to investigate the problem using the calculus tools he was learning at the time. This article is the result of his investigations.

Notice that if a closet of width r has a door mounted as in FIGURE 1 (we will call this the standard mounting) then opening (or closing) the door requires $\frac{\pi r^2}{4}$ square feet of floor space be kept clear of obstacles. Adults *might* be able to do this but it can be an onerous task for a teenager.

This would seem to be the end of the story except that a survey of your closets will quickly convince you that the standard mounting is actually relatively rare on closet doors.

Our (admittedly unscientific) survey of all of the closets we have easy access to convinces us that most of the closet doors in the United States which do not use the standard mounting use a bifold mounting which we discusss next.

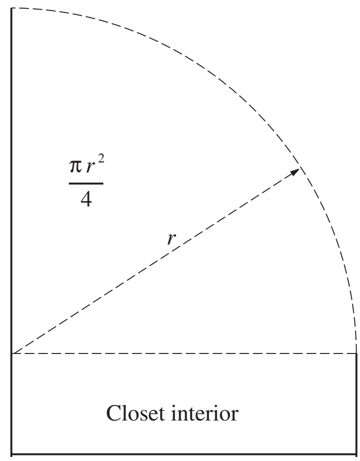

Figure 1 The floor space required for a standard door is the full quarter-circle. A bifold door requires substantially less, but how much less?

Bifold doors

Many closet doors do not have ample room for a standard mounting, therefore to save floor space closet doors are often bifold doors as shown in FIGURE 2. That is, the door is broken and hinged in the middle so that each panel is $r/2$ in length. This allows the door to be mounted in a manner similar to the standard mounting we described earlier except that only the left panel sweeps out a quarter-circle. The inner edge of the left panel of the door is fixed at the point A and the outer edge of the right panel is allowed to slide along the track.

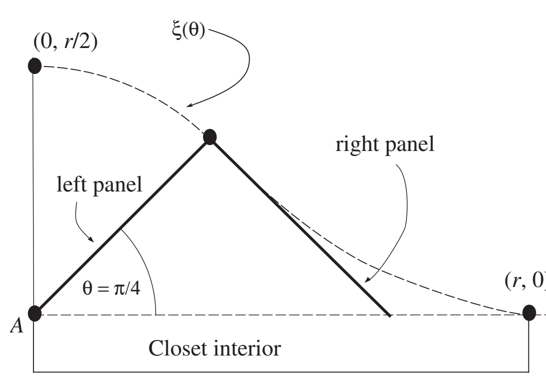

Figure 2 This figure shows the view from above a bifold door as the door closes. It first sweeps out the area under the circular arc of radius $r/2$, but when θ reaches $\pi/4$ the nature of the curve changes. Notice that "bifold" is a misnomer. There is only one fold.

To begin, consider a bifold door starting in the fully open position and closing to the right as in FIGURE 2. The floor space required to close the door is enclosed by the curve we'll call $\xi(\theta)$ and the x and y axes.

It is clear that ξ has two distinct components. The first is simply the circular arc swept out by the left panel of the door as θ proceeds from $\pi/2$ to $\pi/4$. However when $\theta = \pi/4$ the nature of ξ changes. At this point the right panel of the door is tangent to the circular arc. Thus the entire quarter-circle swept out by the left panel of the door is enclosed in the area which has already been swept out. Moreover as the door continues to close, more area outside the quarter-circle continues to be accumulated.

We seek a parameterization of the outer envelope of this area.

To that end, assume that the door is opening as in FIGURE 3 and that $0 \leq \theta \leq \pi/4$. Notice the change here. For the development we are about to present, it is easier to think of the door as opening rather than closing. In either case $\xi(\theta)$ is unchanged.

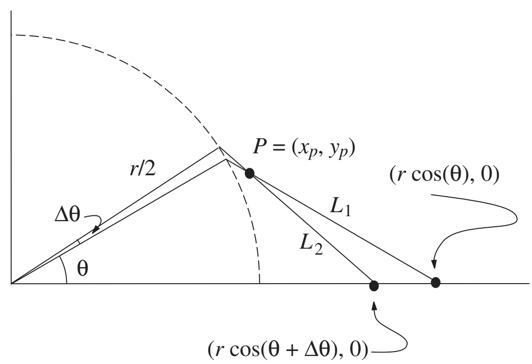

Figure 3 When θ is incremented by $\Delta\theta$ the original position of the right panel of the door and its new position will intersect. The point of intersection at P gives an approximate parameterization of the curve $\xi(\theta)$. As $\Delta\theta \to 0$ this becomes exact.

We increment θ by $\Delta\theta$ and consider the position of the right panel of the door at θ and $\theta + \Delta\theta$. If we can find the coordinates of the point P we have an approximate parameterization of the curve ξ. If $P = (x_P, y_P)$ for a fixed $\Delta\theta$ then it is clear that

$$\xi(\theta) = \lim_{\Delta\theta \to 0} (x_P, y_P)$$

is the parameterization we seek.

Let L_1 be the line of the right panel at θ and let L_2 be the line at $\theta + \Delta\theta$ and observe that the slopes of L_1 and L_2 are $-\tan\theta$ and $-\tan(\theta + \Delta\theta)$, respectively. Thus the equation of L_1 is

$$y = -\tan\theta(x - r\cos\theta) \tag{1}$$

and the equation of L_2 is

$$y = -\tan(\theta + \Delta\theta)(x - r\cos(\theta + \Delta\theta)). \tag{2}$$

Combining equations 1 and 2 we get

$$x_P = r\left(\frac{\sin(\theta + \Delta\theta) - \sin\theta}{\tan(\theta + \Delta\theta) - \tan(\theta)}\right).$$

Putting this back into either L_1 or L_2 gives

$$y_P = -r\tan\theta\left(\frac{\sin(\theta + \Delta\theta) - \sin\theta}{\tan(\theta + \Delta\theta) - \tan(\theta)} - \cos\theta\right).$$

Taking the limit as $\Delta\theta \to 0$ we get

$$
\begin{aligned}
x &= \lim_{\Delta\theta\to 0} r\left(\frac{\sin(\theta+\Delta\theta)-\sin\theta}{\tan(\theta+\Delta\theta)-\tan(\theta)}\right)\\
&= r\lim_{\Delta\theta\to 0}\left(\frac{\frac{\sin(\theta+\Delta\theta)-\sin\theta}{\Delta\theta}}{\frac{\tan(\theta+\Delta\theta)-\tan(\theta)}{\Delta\theta}}\right).
\end{aligned}
$$

Observe that the numerator and denominator of the formula above are just the derivatives of $\sin\theta$ and $\tan\theta$ respectively. Thus

$$x(\theta) = r\frac{\cos\theta}{\sec^2\theta} = r\cos^3\theta.$$

Similarly

$$y(\theta) = r\sin^3\theta.$$

Thus a parameterization for the outer envelope of the floor space used by a bifold door is given by:

$$\xi(\theta) = \begin{cases} \begin{pmatrix} r\cos^3\theta \\ r\sin^3\theta \end{pmatrix} & \text{if } 0 \le \theta \le \pi/4 \\ \begin{pmatrix} r/2\cos\theta \\ r/2\sin\theta \end{pmatrix} & \text{if } \pi/4 \le \theta \le \pi/2 \end{cases}. \tag{3}$$

Notice that letting θ move from 0 to $\pi/2$ opens the door while letting θ move from $\pi/2$ to 0 closes it. To compute the area of the floor space required we need to ensure that we integrate from left to right. The area of the floor space is then given by

$$
\begin{aligned}
\int_{\theta=\pi/2}^{\theta=0} y(\theta)\,\mathrm{d}x &= \int_{\theta=\pi/4}^{\theta=0} r\sin^3(\theta)\frac{\mathrm{d}x}{\mathrm{d}\theta}\,\mathrm{d}\theta + \int_{\theta=\pi/2}^{\theta=\pi/4} r/2\sin\theta\frac{\mathrm{d}x}{\mathrm{d}\theta}\,\mathrm{d}\theta\\
&= 3r^2\int_0^{\pi/4}\sin^4\theta\cos^2\theta\,\mathrm{d}\theta + r^2/4\int_{\pi/4}^{\pi/2}\sin^2\theta\,\mathrm{d}\theta\\
&= \frac{5\pi r^2}{64}.
\end{aligned}
$$

So our initial question is resolved. If a closet r feet wide is covered by a bifold door $\frac{5\pi r^2}{64}$ square feet of floor space is required to accommodate the door. If the same closet is closed with an ordinary door then $\pi r^2/4$ square feet are needed—a savings of nearly 70%.

Adding door panels

It is clear that adding 2, 3, 4, or n folds will reduce the floor space required even further. FIGURE 4 shows the situation with 2 folds. If the doors are hinged so that the angles denoted by θ in FIGURE 4 are always equal then the problem can be approached in the same manner as before as we now show.

As before we perturb θ by $\Delta\theta$ and consider the point of intersection of the rightmost panels in FIGURE 4. In that case the equation of L_1 is (again):

$$y = -\tan\theta(x - r\cos\theta)$$

and the equation of L_2 is (again):

$$y = -\tan(\theta + \Delta\theta)(x - r\cos(\theta + \Delta\theta)).$$

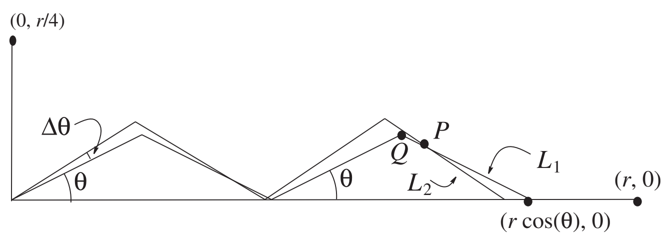

Figure 4 Each door panel has length $r/4$.

Since these are *exactly* the same equations we found in the previous section it follows that the parameterization we seek is (again) the astroid:

$$\begin{pmatrix} r\cos^3\theta \\ r\sin^3\theta \end{pmatrix}.$$

Indeed it should be clear from the above that adding more hinges has no effect on the astroidal portion of the curve. The very same astroid appears regardless of the number of folds in the door as long as all of the panels are hinged so that they make the same angle with the front of the closet (the angle θ in FIGURE 4). This assumption is critical. If the angles are allowed to differ the problem becomes considerably more complex.

Recall however that the curve $\xi(\theta)$ from the previous section had two components. The other portion was the circular arc traced out by the point corresponding to Q in FIGURE 4. To find the corresponding portion for the current curve, which we'll denote by $\xi_2(\theta)$, we need to parameterize the coordinates of the point Q.

Referring again to FIGURE 4 it is clear that

$$Q(\theta) = \begin{pmatrix} 3r/4\cos\theta \\ r/4\sin\theta \end{pmatrix}$$

and that the transition between the components of the curve occurs when $P(\theta) = Q(\theta)$, or when $\theta = \pi/6$. Thus when we have two folds in our door the curve $\xi_2(\theta)$ is

$$\xi_2(\theta) = \begin{cases} \begin{pmatrix} r\cos^3\theta \\ r\sin^3\theta \end{pmatrix}, & \text{if } 0 \le \theta \le \pi/6 \\ \begin{pmatrix} 3r/4\cos\theta \\ r/4\sin\theta \end{pmatrix}, & \text{if } \pi/6 \le \theta \le \pi/2 \end{cases}$$

and in the general case, with n folds, the curve is

$$\xi_n(\theta) = \begin{cases} \begin{pmatrix} r\cos^3\theta \\ r\sin^3\theta \end{pmatrix}, & \text{if } 0 \le \theta \le \cos^{-1}\left(\frac{2n-1}{2n}\right) \\ \begin{pmatrix} \frac{(2n-1)r}{2n}\cos\theta \\ \frac{r}{2n}\sin\theta \end{pmatrix}, & \text{if } \cos^{-1}\left(\frac{2n-1}{2n}\right) \le \theta \le \pi/2 \end{cases}.$$

It seems curious that the same curves, an ellipse and an astroid, appear regardless of how many panels we split our door into. Indeed, the same curves appear in the two-fold (four panel) case even if the panels are of two distinct sizes.

By now the calculation is very familiar so we will not belabor it. Consider the arrangement depicted in FIGURE 5. Again we have two folds (four panels) but they are no longer the same length and we have normalized the sum of the lengths of the panels to 1. If we perturb the angle θ by $\Delta\theta$ and find the intersection point P (not shown in the figure) between the original location of the rightmost panel and its perturbed location we find that the equations of L_1 and L_2 are again precisely the same as in our first problem. Thus the astroid emerges exactly as before. Moreover it is easy to show that a parameterization of the point Q in the figure is:

$$Q(\theta) = \begin{pmatrix} (1/2+\alpha)\cos\theta \\ (1/2-\alpha)\sin\theta \end{pmatrix}$$

which is again an ellipse.

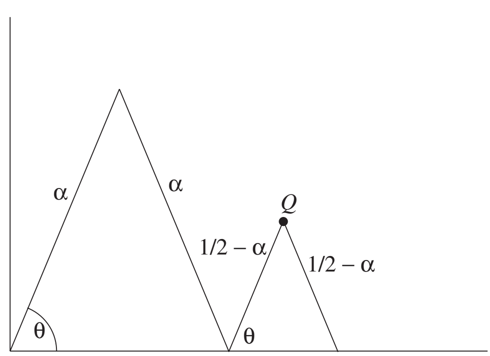

Figure 5 A bifold door with different length panels also generates an astroid and ellipse.

It seems very odd that the same curves keep emerging no matter how we try to generalize the problem.

Wiles' light switch

Andrew Wiles [**5**] has likened mathematics research to walking into an unlighted room. At first all is dark. As you fumble around you begin to get a sense of the location of the objects in the room and the relationships between them. Eventually, if you are lucky, you find the light switch and flip it. Then you see all of the structures and the relationships between them that you were already familiar with as well as new ones that you were only dimly aware of or may not have known at all.

In this section we will flip the light switch for this problem. It turns out that the relationship between ellipses and astroids, which has been our common theme, has been known since antiquity. Archimedes used it to create a mechanical device, known as the "Trammel of Archimedes," for drawing ellipses (see [**1, 2, 3**]).

Consider the following alternate construction of the astroid [**4, 6**]. We begin with vertical line segment whose endpoints are at (0, 1) and the origin (see FIGURE 6). Keeping the length of the segment constant we move the endpoints vertically toward the origin and horizontally toward (1, 0), respectively. The outer envelope of the region thus constructed is the astroid.

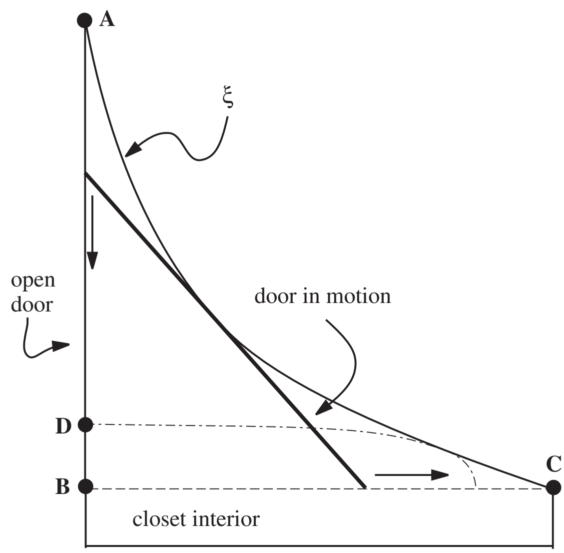

Figure 6 As the door closes points *A* and *B* move toward points *B* and *C*, respectively.

Rather, it is one quarter of the classical astroid of antiquity. This is the portion we have seen so far. If we continue in the same fashion—moving the left end of our line segment vertically to (0, −1) and the right end horizontally back to the origin—we will generate the same curve reflected about the x-axis. Continuing in the same vein until our line segment has returned to its original position gives the full astroid of antiquity as it was known to Archimedes. This is shown in FIGURE 7.

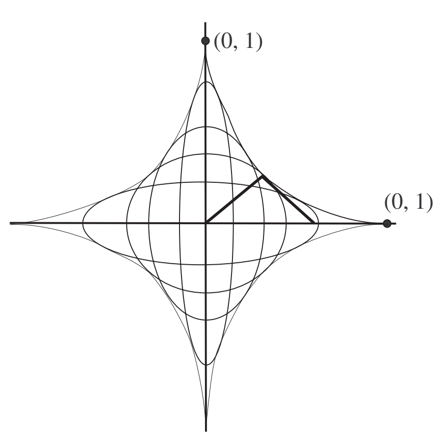

Figure 7 One way to define the astroid is as the outer envelope of a particular set of ellipses as shown here. A bifold door is shown schematically in the first quadrant.

FIGURE 7 also displays the astroid as the outer envelope of a particular set of ellipses. If we fix a point D on our original line segment (see FIGURE 6) between (0, 1) and the origin and follow it as we trace out the astroid then it is easy to show that the path it follows is an ellipse.

It seems that as our folding doors close the peak of the rightmost fold (the point Q seen in figures 4 and 5) traces out one of these ellipses (the elliptical portion of the curve $\xi_n(\theta)$) until it touches the astroid. At that point the rightmost panel of the door is tangent to both the ellipse and the astroid and our $\xi_n(\theta)$ switches modes and begins to follow the astroid.

You can't have a light without a dark to stick it in.

—Arlo Guthrie (1947–)

REFERENCES

1. `http://personal.atl.bellsouth.net/e/d/edwin222/ellipsedevice.htm`.
2. Xah Lee. `http://www.xahlee.org/specialplanecurves_dir/astroid_dir/astroid.html`.
3. Xah Lee. `http://www.xahlee.org/specialplanecurves_dir/ellipse_dir/ellipse.html`.
4. E. H. Lockwood, *A Book of Curves*, Cambridge University Press, 1963.
5. Andrew Wiles, The Proof, Nova, Adventures in Science, 1997, Produced by John Lynch, A co-production of BBC TV and WGBH Boston, ISBN: 1-57807-072-4.
6. Robert C. Yates, *Curves and Their Properties*, National Council of Teachers of Mathematics, 1952.